SOLUTIONS TO EXERCISES OF FIBER BUNDLEFROM LIANG'S TEXTBOOK: DIFFERENTIAL GEOMETRY AND GENERAL RELATIVITY

HAIXUAN LIN

ABSTRACT. There is a well-known textbook in Chinese theoretic physics community called *Differential Geometry and General Relativity* by Liang and Zhou [?] and in Volumn III of the series, there is a chapter on Fiber Bundle. This chapter is a good introduction to the concept of Fiber Bundle for us as beginners and is a good supplement to the book. Here I provide the solutions to the exercises in the chapter.

1. Definitions and Theorems

Definition 1.1. *Principal fiber bundle* consists of a *bundle manifold* P, a *base manifold* M and a *structure Lie group* G. They satisfy the following conditions:

1. G acts freely on P from the right via the action $R : P \times G \to P$;
2. There exists a smooth projection map $\pi : P \to M$ such that for $\forall p \in P$, the fiber over $\pi(p)$ is given by $\pi^{-1}[\pi(p)] = \{pg \mid g \in G\}$;
3. $\forall x \in M$, there exists an open neighborhood U of x in M and a diffeomorphism $T_U : \pi^{-1}[U] \to U \times G$, $T_U(p) = (\pi(p), S_U(p))$, $\forall p \in \pi^{-1}[U]$, where $S_U : \pi^{-1}[U] \to G$ satisfies $S_U(pg) = S_U(p)\, g$, $\forall g \in G$.

Definition 1.2. Let $R_p : G \to P$, $R_p(g) \coloneqq R_g(p)$, then we can define a fundamental vector field

The definition of a *connection* on a principal fiber bundle has three equivalent forms:

Definition 1.3. A connection on a principal fiber bundle $P(M, G)$ assigns to each point $p \in P$ a *horizontal subspace* $H_p \subseteq T_pP$ such that:

1. $T_pP = V_p \oplus H_p$, $\forall p \in P$, where $V_p \coloneqq \{X \in T_pP \mid \pi_*(X) = 0\}$ is *vertical subspace;*
2. $R_{g*}[H_p] = H_{pg}$, $\forall p \in P$, $\forall g \in G$;
3. H_p is C^∞ with respect to p.

Definition 1.4. A connection on a principal fiber bundle $P(M, G)$ is a C^∞ $\mathfrak{g}$-valued 1-form on P, denoted by $\tilde{\omega}$, such that:

1. $\tilde{\omega}_p\left(A^*_p\right) = A$, $\forall A \in \mathfrak{g}$, $\forall p \in P$;

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2. Problem 1

Problem 2.1. Prove Eq.(I-1-4), namely $S_U : \pi^{=1}[x] \to G$ and $R_{\check{p}_U} : G \to \pi^{-1}[x]$ are a pair of inverse mappings.

Proof. $\square$

3. Problem 2

Problem 3.1. Prove Eq.(I-1-5), namely $R_g \circ R_p = R_{pf}$ for $\forall p \in P$, $\forall g \in G$.

Proof. $\square$